The small Dowker space problem

Winter School in Abstract Analysis 2023
section Set Theory Topology seminar

Roy Shalev, *Bar-Ilan University*

Plan for today

I'll be reporting on the following papers:

[48] Assaf Rinot and R.S., *A guessing principle from a Souslin tree, with applications to topology*, accepted to Topology Appl.

[54] Assaf Rinot, R.S and Stevo Todorčević, *A new small Dowker space*, submitted April 2022.

Motivation

A standard question in mathematics asks what properties are preserved under taking products.

Motivation

A standard question in mathematics asks what properties are preserved under taking products.

- The product of two regular topological spaces is regular.

Motivation

A standard question in mathematics asks what properties are preserved under taking products.

- The product of two regular topological spaces is regular.

In contrast, the Sorgenfrey line $\mathbb{R}_l$ is a regular Lindelöf (hence normal) space whose square is not normal (hence, non-Lindelöf).

Motivation

The property of a space $\mathbb{X}$ called binormal, that the product of $\mathbb{X}$ with the closed unit interval be normal, had long been a standard hypothesis for certain homotopy extension theorems.

Motivation

The property of a space $\mathbb{X}$ called binormal, that the product of $\mathbb{X}$ with the closed unit interval be normal, had long been a standard hypothesis for certain homotopy extension theorems.

This raise the question:

Motivation

The property of a space $\mathbb{X}$ called binormal, that the product of $\mathbb{X}$ with the closed unit interval be normal, had long been a standard hypothesis for certain homotopy extension theorems.

This raise the question:

Question (C. H. Dowker, 1951)

Is there a normal topological space whose product with the unit interval is not normal?

Such a space is called Dowker.

The Dowker space problem

Theorem (C. H. Dowker, 1951)

A normal space $\mathbb{X}$ is Dowker iff there exists a $\subseteq$-decreasing sequence $\langle D_n \mid n < \omega \rangle$ of closed sets s.t.:

1. $\bigcap_{n<\omega} D_n = \emptyset$;
2. *if, for every $n < \omega$, U_n is some open set covering D_n, then* $\bigcap_{n<\omega} U_n \neq \emptyset$.

The Dowker space problem (cont.)

Theorem (M. E. Rudin, 1955)

If there is a Souslin tree, then there is a Dowker space of size $\aleph_1$.

The Dowker space problem (cont.)

Theorem (M. E. Rudin, 1955)

If there is a Souslin tree, then there is a Dowker space of size $\aleph_1$.

Curiously, the existence of a Souslin tree was only shown to be consistent around 1967.

The Dowker space problem (cont.)

Theorem (M. E. Rudin, 1955)

If there is a Souslin tree, then there is a Dowker space of size $\aleph_1$.

Curiously, the existence of a Souslin tree was only shown to be consistent around 1967.

Does the existence of a Dowker space follow from ZFC?

The Dowker space problem (cont.)

Theorem (M. E. Rudin, 1972)

There exists a Dowker space of size $(\aleph_{\omega+1})^{\aleph_0}$.

https://yewtu.be/TL-QWMr7-9E

The Dowker space problem (cont.)

Theorem (Balogh, 1996)

There exists a Dowker space of size $2^{\aleph_0}$.

The Dowker space problem (cont.)

Theorem (Balogh, 1996)

There exists a Dowker space of size $2^{\aleph_0}$.

Theorem (Kojman-Shelah, 1998)

There exists a Dowker space of size $\aleph_{\omega+1}$.

The Dowker space problem (cont.)

Theorem (Balogh, 1996)

There exists a Dowker space of size $2^{\aleph_0}$.

Theorem (Kojman-Shelah, 1998)

There exists a Dowker space of size $\aleph_{\omega+1}$.

Whose space is actually smaller?

The Dowker space problem (cont.)

Theorem (Balogh, 1996)

There exists a Dowker space of size $2^{\aleph_0}$.

Theorem (Kojman-Shelah, 1998)

There exists a Dowker space of size $\aleph_{\omega+1}$.

Whose space is actually smaller?

Conjecture (M. E. Rudin, 1990)

There exists a Dowker space of size $\aleph_1$.

The small Dowker space problem

The list of known sufficient conditions for the existence of a Dowker space of size $\aleph_1$ include

The small Dowker space problem

The list of known sufficient conditions for the existence of a Dowker space of size $\aleph_1$ include CH (Juhász, Kunen and Rudin, 1976),

The small Dowker space problem

The list of known sufficient conditions for the existence of a Dowker space of size $\aleph_1$ include CH (Juhász, Kunen and Rudin, 1976), ♣ (de Caux, 1977),

The small Dowker space problem

The list of known sufficient conditions for the existence of a Dowker space of size $\aleph_1$ include CH (Juhász, Kunen and Rudin, 1976), $\clubsuit$ (de Caux, 1977), a Luzin set (Todorčević, 1989),

The small Dowker space problem

The list of known sufficient conditions for the existence of a Dowker space of size $\aleph_1$ include CH (Juhász, Kunen and Rudin, 1976), ♣ (de Caux, 1977), a Luzin set (Todorčević, 1989), and a certain tailored instance of a strong club-guessing principle (Hernńdez-Hernńdez and Szeptycki, 2009).

The small Dowker space problem

The list of known sufficient conditions for the existence of a Dowker space of size $\aleph_1$ include CH (Juhász, Kunen and Rudin, 1976), $\clubsuit$ (de Caux, 1977), a Luzin set (Todorčević, 1989), and a certain tailored instance of a strong club-guessing principle (Hernńdez-Hernńdez and Szeptycki, 2009).

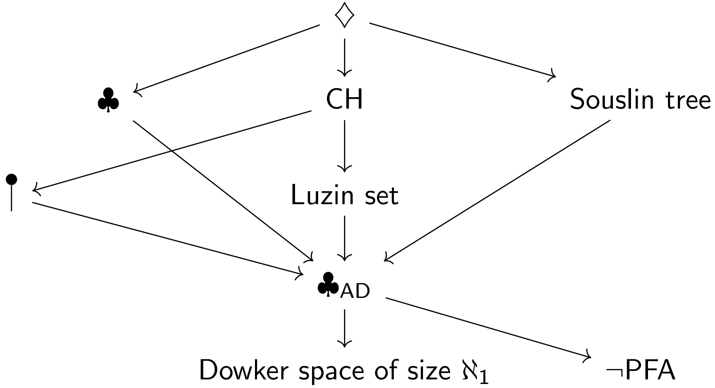
In [54], we present a new sufficient condition, namely, the following weakening of the continuum hypothesis:

Definition (Broverman-Ginsburg-Kunen-Tall, 1978)

$\pin$ asserts there is a list $\langle A_\alpha \mid \alpha < \aleph_1\rangle$ of infinite subsets of $\aleph_1$ such that for every uncountable $B \subseteq \aleph_1$, there is $\alpha < \omega_1$ with $A_\alpha \subseteq B$.

Diagram of implications

Along the way, we unify the above-mentioned results, factoring the Dowker space constructions through a new ‘guessing’ principle that we call $\clubsuit_{\mathrm{AD}}$.



We denote by L the set of all nonzero limit countable ordinals. For every infinite ordinal $\beta < \omega_1$, we denote by α_β the unique $\alpha \in \mathsf{L}$ such that $\alpha \leq \beta < \alpha + \omega$.

We denote by L the set of all nonzero limit countable ordinals. For every infinite ordinal $\beta < \omega_1$, we denote by α_β the unique $\alpha \in \mathsf{L}$ such that $\alpha \leq \beta < \alpha + \omega$.

Definition ([48])

$\clubsuit_{AD}$ asserts there is a matrix $\langle A_{\alpha,n} \mid \alpha \in \mathsf{L}, n < \omega \rangle$ such that:

0. for every $\alpha \in \mathsf{L}$, $\langle A_{\alpha,n} \mid n < \omega \rangle$ consists of pairwise disjoint cofinal subsets of α;

$\clubsuit_{\mathrm{AD}}$

We denote by L the set of all nonzero limit countable ordinals. For every infinite ordinal $\beta < \omega_1$, we denote by α_β the unique $\alpha \in \mathrm{L}$ such that $\alpha \leq \beta < \alpha + \omega$.

Definition ([48])

$\clubsuit_{\mathrm{AD}}$ asserts there is a matrix $\langle A_{\alpha,n} \mid \alpha \in \mathrm{L}, n < \omega \rangle$ such that:

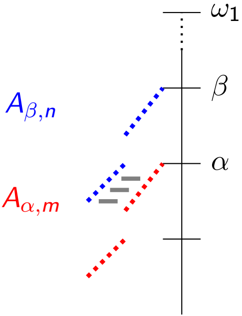
0. for every $\alpha \in \mathrm{L}$, $\langle A_{\alpha,n} \mid n < \omega \rangle$ consists of pairwise disjoint cofinal subsets of α;
1. for every uncountable $B \subseteq \aleph_1$, there is $\alpha \in \mathrm{L}$ such that $\sup(A_{\alpha,n} \cap B) = \alpha$ for all $n < \omega$;

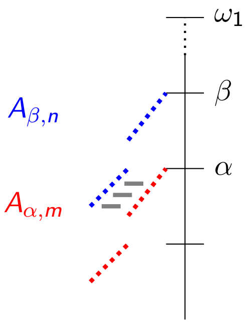
We denote by L the set of all nonzero limit countable ordinals. For every infinite ordinal $\beta < \omega_1$, we denote by α_β the unique $\alpha \in \mathsf{L}$ such that $\alpha \leq \beta < \alpha + \omega$.

Definition ([48])

$\clubsuit_{\mathsf{AD}}$ asserts there is a matrix $\langle A_{\alpha,n} \mid \alpha \in \mathsf{L}, n < \omega \rangle$ such that:

0. for every $\alpha \in \mathsf{L}$, $\langle A_{\alpha,n} \mid n < \omega \rangle$ consists of pairwise disjoint cofinal subsets of α;
1. for every uncountable $B \subseteq \aleph_1$, there is $\alpha \in \mathsf{L}$ such that $\sup(A_{\alpha,n} \cap B) = \alpha$ for all $n < \omega$;
2. for all $(\alpha, n) \neq (\beta, m)$, $\sup(A_{\alpha,n} \cap A_{\beta,m}) < \alpha$.


ω_1
β
$A_{\beta,n}$
α
$A_{\alpha,m}$



Disjointifying initial segments

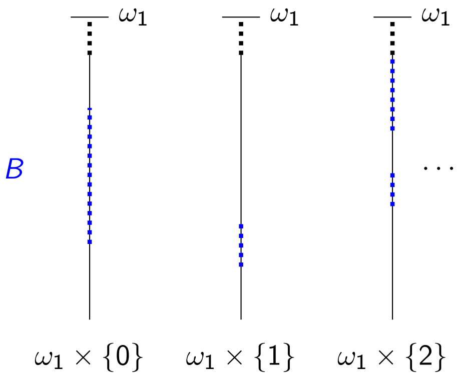
For every $\epsilon < \aleph_1$, there exists a map $f : (\mathsf{L} \cap \epsilon) \times \omega \to \epsilon$ such that

1. $f(\alpha, n) < \alpha$;
2. $\{A_{\alpha,n} \setminus f(\alpha, n) \mid (\alpha, n) \in \mathrm{dom}(f)\}$ is a pairwise disjoint family.

Constructing the space

A few promises

Our space $\mathbb{X} = (X, \tau)$ will have underlying set $\omega_1 \times \omega$.
For all $B \subseteq X$ and $j < \omega$, we write $\pi_j(B) := \{\xi < \omega_1 \mid (\xi, j) \in B\}$ for its j^{th}-section.



A few promises

Our space $\mathbb{X} = (X, \tau)$ will have underlying set $\omega_1 \times \omega$.

For all $B \subseteq X$ and $j < \omega$, we write $\pi_j(B) := \{\xi < \omega_1 \mid (\xi, j) \in B\}$ for its j^{th}-section.

For each $x \in X$, we shall define a weak neighborhood base $\mathcal{N}_x$, and then a subset $U \subseteq X$ will be declared to be τ-open iff for every $x \in U$ there is $N \in \mathcal{N}_x$ with $N \subseteq U$.

A few promises

Our space $\mathbb{X} = (X, \tau)$ will have underlying set $\omega_1 \times \omega$.
For all $B \subseteq X$ and $j < \omega$, we write $\pi_j(B) := \{\xi < \omega_1 \mid (\xi, j) \in B\}$ for its j^{th}-section.
For each $x \in X$, we shall define a weak neighborhood base $\mathcal{N}_x$, and then a subset $U \subseteq X$ will be declared to be τ-open iff for every $x \in U$ there is $N \in \mathcal{N}_x$ with $N \subseteq U$.
All $x \in \omega \times \omega$ will be isolated (so $\mathcal{N}_x = \{\{x\}\}$).

A few promises

Our space $\mathbb{X} = (X, \tau)$ will have underlying set $\omega_1 \times \omega$.
For all $B \subseteq X$ and $j < \omega$, we write $\pi_j(B) := \{\xi < \omega_1 \mid (\xi, j) \in B\}$ for its j^{th}-section.
For each $x \in X$, we shall define a weak neighborhood base $\mathcal{N}_x$, and then a subset $U \subseteq X$ will be declared to be τ-open iff for every $x \in U$ there is $N \in \mathcal{N}_x$ with $N \subseteq U$.
All $x \in \omega \times \omega$ will be isolated (so $\mathcal{N}_x = \{\{x\}\}$).
For every $x = (\beta, n)$ in $X \setminus (\omega \times \omega)$, each $N \in \mathcal{N}_x$ will be a subset of $(\beta + 1) \times (n + 1)$.

A few promises

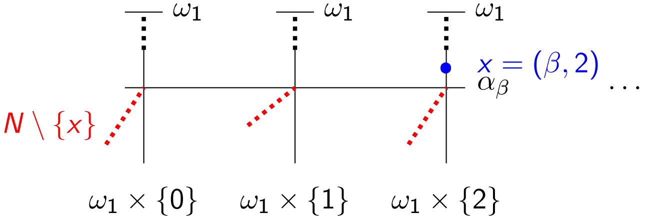
Our space $\mathbb{X} = (X, \tau)$ will have underlying set $\omega_1 \times \omega$.

For all $B \subseteq X$ and $j < \omega$, we write $\pi_j(B) := \{\xi < \omega_1 \mid (\xi, j) \in B\}$ for its j^{th}-section.

For each $x \in X$, we shall define a weak neighborhood base $\mathcal{N}_x$, and then a subset $U \subseteq X$ will be declared to be τ-open iff for every $x \in U$ there is $N \in \mathcal{N}_x$ with $N \subseteq U$.

All $x \in \omega \times \omega$ will be isolated (so $\mathcal{N}_x = \{\{x\}\}$).

For every $x = (\beta, n)$ in $X \setminus (\omega \times \omega)$, each $N \in \mathcal{N}_x$ will be a subset of $(\beta + 1) \times (n + 1)$, and for every $j \leq n$, $\pi_j(N \setminus \{x\})$ will be a cofinal subset of α_β.



A few promises

Our space $\mathbb{X} = (X, \tau)$ will have underlying set $\omega_1 \times \omega$.
For all $B \subseteq X$ and $j < \omega$, we write $\pi_j(B) := \{\xi < \omega_1 \mid (\xi, j) \in B\}$ for its j^{th}-section.
For each $x \in X$, we shall define a weak neighborhood base $\mathcal{N}_x$, and then a subset $U \subseteq X$ will be declared to be τ-open iff for every $x \in U$ there is $N \in \mathcal{N}_x$ with $N \subseteq U$.
All $x \in \omega \times \omega$ will be isolated (so $\mathcal{N}_x = \{\{x\}\}$).
For every $x = (\beta, n)$ in $X \setminus (\omega \times \omega)$, each $N \in \mathcal{N}_x$ will be a subset of $(\beta + 1) \times (n + 1)$, and for every $j \leq n$, $\pi_j(N \setminus \{x\})$ will be a cofinal subset of α_β.

Consequence 1

For all $\delta < \omega_1$ and $n < \omega$, $\delta \times n$ is τ-open, and $D_n := \omega_1 \times (\omega \setminus n)$ is τ-closed. $\langle D_n \mid n < \omega \rangle$ is $\subseteq$-decreasing, and $\bigcap_{n<\omega} D_n = \emptyset$.
The first part implies that $\mathbb{X}$ is not Lindelöf.

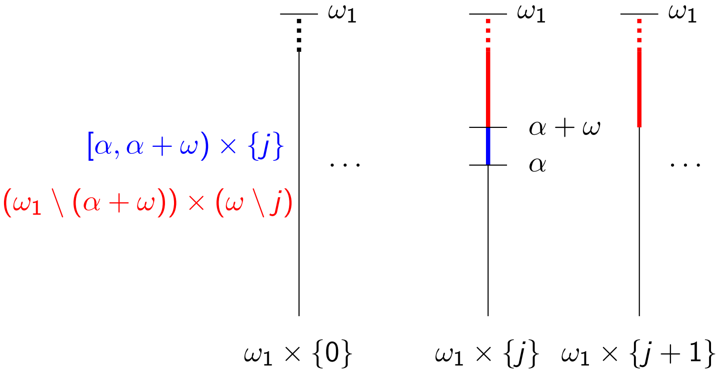
A few promises

Our space $\mathbb{X} = (X, \tau)$ will have underlying set $\omega_1 \times \omega$.
For all $B \subseteq X$ and $j < \omega$, we write $\pi_j(B) := \{\xi < \omega_1 \mid (\xi, j) \in B\}$ for its j^{th}-section.
For each $x \in X$, we shall define a weak neighborhood base $\mathcal{N}_x$, and then a subset $U \subseteq X$ will be declared to be τ-open iff for every $x \in U$ there is $N \in \mathcal{N}_x$ with $N \subseteq U$.
All $x \in \omega \times \omega$ will be isolated (so $\mathcal{N}_x = \{\{x\}\}$).
For every $x = (\beta, n)$ in $X \setminus (\omega \times \omega)$, each $N \in \mathcal{N}_x$ will be a subset of $(\beta + 1) \times (n + 1)$, and for every $j \leq n$, $\pi_j(N \setminus \{x\})$ will be a cofinal subset of α_β.

Consequence 2: The domino effect

For every $(\alpha, j) \in \mathsf{L} \times \omega$, the τ-closure of the strip $[\alpha, \alpha + \omega) \times \{j\}$ covers the following tail of ω_1 times a tail of ω:

$$(\omega_1 \setminus (\alpha + \omega)) \times (\omega \setminus j).$$



ω_1
ω_1
ω_1
$[\alpha, \alpha+\omega) \times \{j\}$
$\cdots$
$\alpha + \omega$
α
$\cdots$
$(\omega_1 \setminus (\alpha+\omega)) \times (\omega \setminus j)$
$\omega_1 \times \{0\}$
$\omega_1 \times \{j\}$
$\omega_1 \times \{j+1\}$

The actual construction

Re-index $\langle A_{\alpha,n} \mid \alpha \in \mathsf{L}, n < \omega \rangle$ as $\langle A^j_{\beta,n} \mid \omega \leq \beta < \omega_1, j \leq n < \omega \rangle$ such that, for every $\alpha \in \mathsf{L}$,

$$\{A_{\alpha,n} \mid n < \omega\} \stackrel{1-1}{=\!=} \{A^j_{\beta,n} \mid \alpha \leq \beta < \alpha + \omega, j \leq n < \omega\}.$$

The actual construction

Re-index $\langle A_{\alpha,n} \mid \alpha \in \mathsf{L}, n < \omega \rangle$ as $\langle A^j_{\beta,n} \mid \omega \leq \beta < \omega_1, j \leq n < \omega \rangle$ such that, for every $\alpha \in \mathsf{L}$,

$$\{A_{\alpha,n} \mid n < \omega\} \overset{1-1}{=\!=} \{A^j_{\beta,n} \mid \alpha \leq \beta < \alpha + \omega, j \leq n < \omega\}.$$

So, each $A^j_{\beta,n}$ is a cofinal subset of α_β.

The actual construction

Re-index $\langle A_{\alpha,n} \mid \alpha \in \mathsf{L}, n < \omega \rangle$ as $\langle A^j_{\beta,n} \mid \omega \le \beta < \omega_1, j \le n < \omega \rangle$ such that, for every $\alpha \in \mathsf{L}$,

$$\{A_{\alpha,n} \mid n < \omega\} \overset{1-1}{=\!=} \{A^j_{\beta,n} \mid \alpha \le \beta < \alpha + \omega, j \le n < \omega\}.$$

So, each $A^j_{\beta,n}$ is a cofinal subset of α_β.

- For $x \in \omega \times \omega$, let $\mathcal{N}_x = \{\{x\}\}$.
- For $x = (\beta, n)$ in $X \setminus \omega \times \omega$, let $\mathcal{N}_x := \{N^\epsilon_x \mid \epsilon < \alpha_\beta\}$, where

$$N^\epsilon_x := \{x\} \cup \bigcup_{j \le n} ((A^j_{\beta,n} \setminus \epsilon) \times \{j\}).$$

The actual construction

Re-index $\langle A_{\alpha,n} \mid \alpha \in \mathsf{L}, n < \omega\rangle$ as $\langle A^j_{\beta,n} \mid \omega \leq \beta < \omega_1, j \leq n < \omega\rangle$ such that, for every $\alpha \in \mathsf{L}$,

$$\{A_{\alpha,n} \mid n < \omega\} \overset{1-1}{=\!=} \{A^j_{\beta,n} \mid \alpha \leq \beta < \alpha + \omega, j \leq n < \omega\}.$$

So, each $A^j_{\beta,n}$ is a cofinal subset of α_β.

- For $x \in \omega \times \omega$, let $\mathcal{N}_x = \{\{x\}\}$.
- For $x = (\beta, n)$ in $X \setminus \omega \times \omega$, let $\mathcal{N}_x := \{N^\epsilon_x \mid \epsilon < \alpha_\beta\}$, where

$$N^\epsilon_x := \{x\} \cup \bigcup_{j \leq n} ((A^j_{\beta,n} \setminus \epsilon) \times \{j\}).$$

*The epsilons are there to ensure that the outcome space $\mathbb{X}$ is T_1.
Indeed, $\bigcap \mathcal{N}_x = \{x\}$.

The actual construction

Re-index $\langle A_{\alpha,n} \mid \alpha \in \mathsf{L}, n < \omega\rangle$ as $\langle A^j_{\beta,n} \mid \omega \leq \beta < \omega_1, j \leq n < \omega\rangle$ such that, for every $\alpha \in \mathsf{L}$,

$$\{A_{\alpha,n} \mid n < \omega\} \stackrel{1-1}{=\!=} \{A^j_{\beta,n} \mid \alpha \leq \beta < \alpha + \omega, j \leq n < \omega\}.$$

So, each $A^j_{\beta,n}$ is a cofinal subset of α_β.

- For $x \in \omega \times \omega$, let $\mathcal{N}_x = \{\{x\}\}$.
- For $x = (\beta, n)$ in $X \setminus \omega \times \omega$, let $\mathcal{N}_x := \{N^\epsilon_x \mid \epsilon < \alpha_\beta\}$, where

$$N^\epsilon_x := \{x\} \cup \bigcup_{j \leq n}((A^j_{\beta,n} \setminus \epsilon) \times \{j\}).$$

and $\pi_j(N^\epsilon_x \setminus \{x\}) = A^j_{\beta,n} \setminus \epsilon$ is a cofinal subset of α_β, as promised.

The actual construction

Re-index $\langle A_{\alpha,n} \mid \alpha \in \mathsf{L}, n < \omega \rangle$ as $\langle A^j_{\beta,n} \mid \omega \leq \beta < \omega_1, j \leq n < \omega \rangle$ such that, for every $\alpha \in \mathsf{L}$,

$$\{A_{\alpha,n} \mid n < \omega\} \overset{1-1}{=\!=} \{A^j_{\beta,n} \mid \alpha \leq \beta < \alpha + \omega, j \leq n < \omega\}.$$

So, each $A^j_{\beta,n}$ is a cofinal subset of α_β.

- For $x \in \omega \times \omega$, let $\mathcal{N}_x = \{\{x\}\}$.
- For $x = (\beta, n)$ in $X \setminus \omega \times \omega$, let $\mathcal{N}_x := \{N^\epsilon_x \mid \epsilon < \alpha_\beta\}$, where

$$N^\epsilon_x := \{x\} \cup \bigcup_{j \leq n} ((A^j_{\beta,n} \setminus \epsilon) \times \{j\}).$$

Consequence 3

Given $(\beta, n) \in X \setminus \omega \times \omega$ and $B \subseteq X$, if there exists $j < \omega$ such that $\sup(A^j_{\beta,n} \cap \pi_j(B)) = \alpha_\beta$, then $(\beta, n) \in \mathrm{cl}(B)$.

Verifications

Lemma
Every τ-closed uncountable $B \subseteq X$ contains a 'tail', i.e., there is $(\gamma, j) \in \mathsf{L} \times \omega$ such that $(\omega_1 \setminus \gamma) \times (\omega \setminus j) \subseteq B$.

Proof. Given an uncountable $B \subseteq X$, find the least $j < \omega$ such that $\pi_j(B)$ is uncountable.

Verifications

Lemma

Every τ-closed uncountable $B \subseteq X$ contains a 'tail', i.e., there is $(\gamma, j) \in \mathsf{L} \times \omega$ such that $(\omega_1 \setminus \gamma) \times (\omega \setminus j) \subseteq B$.

Proof. Given an uncountable $B \subseteq X$, find the least $j < \omega$ such that $\pi_j(B)$ is uncountable. By the guessing feature of $\clubsuit_{\mathrm{AD}}$, pick $\alpha \in \mathsf{L}$ such that $\sup(A_{\alpha,n} \cap \pi_j(B)) = \alpha$ for all $n < \omega$.

Verifications

Lemma

Every τ-closed uncountable $B \subseteq X$ contains a 'tail', i.e., there is $(\gamma, j) \in \mathsf{L} \times \omega$ such that $(\omega_1 \setminus \gamma) \times (\omega \setminus j) \subseteq B$.

Proof. Given an uncountable $B \subseteq X$, find the least $j < \omega$ such that $\pi_j(B)$ is uncountable. By the guessing feature of $\clubsuit_{\mathrm{AD}}$, pick $\alpha \in \mathsf{L}$ such that $\sup(A_{\alpha,n} \cap \pi_j(B)) = \alpha$ for all $n < \omega$.
In particular, $\sup(A^j_{\beta,j} \cap \pi_j(B)) = \alpha$ for all $\beta \in [\alpha, \alpha+\omega)$. So, the τ-closure of the countable set $Y := B \cap (\alpha \times \{j\})$ covers $[\alpha, \alpha+\omega) \times \{j\}$.

Verifications

Lemma

Every τ-closed uncountable $B \subseteq X$ contains a 'tail', i.e., there is $(\gamma, j) \in \mathsf{L} \times \omega$ such that $(\omega_1 \setminus \gamma) \times (\omega \setminus j) \subseteq B$.

Proof. Given an uncountable $B \subseteq X$, find the least $j < \omega$ such that $\pi_j(B)$ is uncountable. By the guessing feature of $\clubsuit_{\mathrm{AD}}$, pick $\alpha \in \mathsf{L}$ such that $\sup(A_{\alpha,n} \cap \pi_j(B)) = \alpha$ for all $n < \omega$.
In particular, $\sup(A^j_{\beta,j} \cap \pi_j(B)) = \alpha$ for all $\beta \in [\alpha, \alpha+\omega)$. So, the τ-closure of the countable set $Y := B \cap (\alpha \times \{j\})$ covers $[\alpha, \alpha+\omega) \times \{j\}$.
By the domino effect, the τ-closure of Y moreover covers

$$(\omega_1 \setminus (\alpha + \omega)) \times (\omega \setminus j). \quad \square$$

Verifications

Lemma

Every τ-closed uncountable $B \subseteq X$ contains a 'tail', i.e., there is $(\gamma, j) \in \mathsf{L} \times \omega$ such that $(\omega_1 \setminus \gamma) \times (\omega \setminus j) \subseteq B$.

Proof. Given an uncountable $B \subseteq X$, find the least $j < \omega$ such that $\pi_j(B)$ is uncountable. By the guessing feature of $\clubsuit_{\mathrm{AD}}$, pick $\alpha \in \mathsf{L}$ such that $\sup(A_{\alpha,n} \cap \pi_j(B)) = \alpha$ for all $n < \omega$.
In particular, $\sup(A^j_{\beta,j} \cap \pi_j(B)) = \alpha$ for all $\beta \in [\alpha, \alpha+\omega)$. So, the τ-closure of the countable set $Y := B \cap (\alpha \times \{j\})$ covers $[\alpha, \alpha+\omega) \times \{j\}$.
By the domino effect, the τ-closure of Y moreover covers

$$(\omega_1 \setminus (\alpha + \omega)) \times (\omega \setminus j). \quad \square$$

The above proof shows that the space is hereditary separable, so altogether $\mathbb{X}$ is an S-space.

Verifications (cont.)

To verify normality, let K_0, K_1 be two disjoint τ-closed subsets of X. As any uncountable closed set contains a 'tail', at least one of the sets must be countable. So, one of these sets is covered by $\epsilon \times \omega$ for some $\epsilon \in \mathrm{L}$. Now, construct two disjoint τ-open sets V_0, V_1 using the feature of disjointifying initial segments.

Verifications (cont.)

To verify normality, let K_0, K_1 be two disjoint τ-closed subsets of X. As any uncountable closed set contains a 'tail', at least one of the sets must be countable. So, one of these sets is covered by $\epsilon \times \omega$ for some $\epsilon \in \mathsf{L}$. Now, construct two disjoint τ-open sets V_0, V_1 using the feature of disjointifying initial segments.

Finally, to prove that $\mathbb{X}$ is Dowker, recall that each $D_n := \omega_1 \times (\omega \setminus n)$ is an uncountable τ-closed set, and that $\bigcap_{n<\omega} D_n = \emptyset$. We need to show that, if, for every $n < \omega$, U_n is some open set covering D_n, then $\bigcap_{n<\omega} U_n \neq \emptyset$.

Verifications (cont.)

To verify normality, let K_0, K_1 be two disjoint τ-closed subsets of X. As any uncountable closed set contains a 'tail', at least one of the sets must be countable. So, one of these sets is covered by $\epsilon \times \omega$ for some $\epsilon \in \mathsf{L}$. Now, construct two disjoint τ-open sets V_0, V_1 using the feature of disjointifying initial segments.

Finally, to prove that $\mathbb{X}$ is Dowker, recall that each $D_n := \omega_1 \times (\omega \setminus n)$ is an uncountable τ-closed set, and that $\bigcap_{n<\omega} D_n = \emptyset$. We need to show that, if, for every $n < \omega$, U_n is some open set covering D_n, then $\bigcap_{n<\omega} U_n \neq \emptyset$.
For each $n < \omega$, $F_n := X \setminus U_n$ is a closed set disjoint from D_n. Since D_n is uncountable, F_n must be countable. So $\bigcup_{n<\omega} F_n$ is countable, and hence $\bigcap_{n<\omega} U_n = X \setminus \bigcup_{n<\omega} F_n$ is nonempty. ■

Thank you!